Coefficients of Recursive Linear Time-Invariant First-Order Low-Pass and High-Pass Filters

(v0.1)

Cliff Sparks
www.arpchord.com

The following is a quick overview of recursive linear time-invariant first-order low-pass and high-pass filters. Such algorithms are often used in simple audio applications, however there are many implementations of these algorithms which use an approximate or incorrect cutoff frequency formula, some that have inaccurate gain compensation, and most with no explanation of where the coefficients in the recursion formula originate from. We derive from first principles formulae for these coefficients for the low-pass and high-pass cases.

Setup

Consider the *difference equation*

$$y_n = b_0 x_n + b_1 x_{n-1} - a_1 y_{n-1}$$

where x_n denotes input data at time sample n and y_n denotes corresponding output data. The coefficients b_0, b_1, a_1 are parameters to be determined. The *transfer function* for the above difference equation is

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} = b_0 \frac{(1-\beta z^{-1})}{(1-\alpha z^{-1})}$$

where α is the *pole* of the transfer function and β is the *zero* of the transfer function. It follows that

$$a_1 = -\alpha, \ \ b_1 = -b_0 \beta .$$

To obtain real output from real input we require b_0, α, β to be real. We also require $|\alpha| < 1$ for filter stability (at $|\alpha| = 1$ the filter is said to be *marginally* stable). Furthermore, we may set $b_0 > 0$ since the sign of b_0 only manifests itself as a π phase shift in the output.

The *gain* $G(\omega)$ of the filter is

$$G(\omega) = |H(\mathrm{e}^{\mathrm{j}\omega})| = b_0 \sqrt{\frac{1-2\beta\cos(\omega)+\beta^2}{1-2\alpha\cos(\omega)+\alpha^2}}.$$

The *phase* $\Theta(\omega)$ of the filter is

$$\Theta(\omega) = \arg(H(\mathrm{e}^{\mathrm{j}\omega})) = \tan^{-1}\left(\frac{\beta\sin(\omega)}{1-\beta\cos(\omega)}\right) - \tan^{-1}\left(\frac{\alpha\sin(\omega)}{1-\alpha\cos(\omega)}\right).$$

The *radian frequency* ω is related to the *normalised frequency* f via $\omega = 2\pi f$.

We now concern ourselves with low-pass and high-pass analysis, and as is the goal with most filters, concentrate on the gain only.

Low-Pass

Firstly set the gain of the filter to be unity at zero frequency (DC). For these first-order filters this will also be the place of maximum gain. Thus

$$1 = G(0) = b_0 \frac{(1-\beta)}{(1-\alpha)}.$$

We can construct three types of low-pass: a *one-zero* filter, a *one-pole* filter, and a *one-pole-one-zero* filter. Consider the one-zero case. Here $\alpha = 0$ and so

$$b_0 = \frac{1}{(1-\beta)},\ \ b_1 = -\frac{\beta}{(1-\beta)},\ \ a_1 = 0.$$

We will choose a β which satisfies $G^2(\omega_c) = 1/2,$ where we call ω_c the *cutoff frequency.* Thus

$$\frac{1}{2} = \frac{(1-2\beta\cos(\omega_c)+\beta^2)}{(1-\beta)^2}$$

Solving this quadratic yields

$$\beta = -1 + 2\cos(\omega_c) + 2\sqrt{(\cos(\omega_c)-1)\cos(\omega_c)}$$

where we choose the positive root for convenience so that β lies inside the unit circle. We see that the range of ω_c is restricted in order for β to be real. Specifically

$$\frac{\pi}{2} \leqslant \omega_c \leqslant \pi$$

with $\omega_c = \pi/2$ yielding the maximum possible overall attenuation. This is a major sonic restriction, however we do have that when $\omega_c = \pi/2,$ $\beta = -1,$ thus frequencies at Nyquist are attenuated all the way to zero.

Let's now examine the one-pole case. Here $\beta = 0$ and so

$$b_0 = (1-\alpha),\ b_1 = 0,\ a_1 = -\alpha.$$

Choose an α which satisfies $G^2(\omega_c) = 1/2$:

$$\frac{1}{2} = \frac{(1-\alpha)^2}{(1-2\alpha\cos(\omega_c)+\alpha^2)}$$

which is a quadratic with solution

$$\alpha = 2 - \cos(\omega_c) - \sqrt{(\cos(\omega_c)-3)(\cos(\omega_c)-1)}$$

where we have chosen the positive root to ensure $|\alpha| \leqslant 1.$ Here there is no restriction on ω_c ; α is always real. This filter also sounds much better than the one-zero version. The one-pole low-pass filter is the most common first-order low-pass filter algorithm. However there are no places where the attenuation drops all the way to zero.

Now for the one-pole-one-zero case. We have $G^2(\omega_c) = 1/2$ but we need another relation between α and β. With higher-order filters conditions such as maximal flatness in the passband or minimal phase distortion in the passband can be set. Such conditions set certain terms in a Taylor series expansion involving ω to zero. The author has tried implementing maximal flatness and minimal phase distortion in the present first-order case and found that only if α and/or β are complex can either be achieved. However, there is a simpler choice of condition: that of ensuring the gain at Nyquist is zero. Setting $G(\pi) = 0$ yields $1+2\beta+\beta^2 = 0$ i.e. $\beta = -1$. Then

$$b_0 = \frac{(1-\alpha)}{2}, \quad b_1 = \frac{(1-\alpha)}{2}, \quad a_1 = -\alpha .$$

Now apply $G^2(\omega_c) = 1/2$:

$$\frac{1}{2} = \frac{(1-\alpha)^2}{4} \frac{2(1+\cos(\omega_c))}{(1-2\alpha\cos(\omega_c)+\alpha^2)}$$

which is a quadratic with solution

$$\alpha = \frac{1-\sin(\omega_c)}{\cos(\omega_c)}$$

choosing the negative root so that α is finite at $\omega_c = \pi/2$. So as for the one-pole case there are no restrictions on ω_c. This one-pole-one-zero version of the low-pass filter sounds the best, as one might expect.

High-Pass

Firstly set the gain of the filter to be unity at maximum frequency (Nyquist). For these first-order filters this will also be the place of maximum gain. Thus

$$1 = G(\pi) = b_0 \frac{(1+\beta)}{(1+\alpha)}.$$

As for the low-pass filter, we can construct three types of high-pass. For the one-zero case $\alpha = 0$ thus

$$b_0 = \frac{1}{(1+\beta)}, \quad b_1 = -\frac{\beta}{(1+\beta)}, \quad a_1 = 0$$

with

$$\beta = 1+2\cos(\omega_c)-2\sqrt{(\cos(\omega_c)+1)\cos(\omega_c)}$$

Thus as for the low-pass filter the range of ω_c is restricted in order for β to be real. Specifically

$$0 \leqslant \omega_c \leqslant \frac{\pi}{2}$$

with $\omega_c = \pi/2$ yielding the maximum possible overall attenuation. This again is a major sonic restriction, however we do have that when $\omega_c = \pi/2$, $\beta = 1$, thus frequencies at DC are attenuated all the way to zero.

Now for the one-pole case. Here $\beta = 0$ thus

$$b_0 = (1+\alpha),\ b_1 = 0,\ a_1 = -\alpha$$

with

$$\alpha = -2-\cos(\omega_c)+\sqrt{(\cos(\omega_c)+3)(\cos(\omega_c)+1)}$$

and no restriction on ω_c. This filter sounds roughly similar to the one-zero version; the difference between the one-pole and one-zero is much less pronounced than for the low-pass case. Again there are no places where the attenuation drops all the way to zero.

Now for the one-pole-one-zero case. Following the low-pass example, let's set the gain at DC to be zero. That is, $G(0) = 0$ yielding $1-2\beta+\beta^2 = 0$ i.e. $\beta = 1$. Then

$$b_0 = \frac{(1+\alpha)}{2},\ b_1 = -\frac{(1+\alpha)}{2},\ a_1 = -\alpha$$

with

$$\alpha = \frac{1-\sin(\omega_c)}{\cos(\omega_c)}$$

which is the same equation as in the low-pass case. Thus again no restriction on ω_c, and also this is the best-sounding version of the three high-pass filters. The one-pole-one-zero high-pass filter is the most common first-order high-pass filter algorithm, where it is often mistakenly labelled as only one-pole.

Cascading

First-order filters can be cascaded, by placing them in series, to yield a type of higher-order filter. The total attenuation (or boost) of any particular frequency is then simply the multiplication of the attenuation (or boost) of that frequency by each of the individual filters. (This is not true, of course, for general higher-order filters involving complex conjugate zeros and poles.) Thus places of unity and zero gain remain unchanged, so we need only reconsider the cutoff frequency condition in the above low-pass and high-pass filters. If m of the one-pole-one-zero low-pass filters are cascaded in series we have

$$\frac{1}{2^{1/m}} = \frac{(1-\alpha)^2}{4}\frac{2(1+\cos(\omega_c))}{(1-2\alpha\cos(\omega_c)+\alpha^2)}$$

with solution

$$\alpha = \frac{2^{1/m}-2(\sqrt{2^{1/m}}-1)\sin(\omega_c)-(2-2^{1/m})\cos(\omega_c)}{2^{1/m}\cos(\omega_c)-(2-2^{1/m})}$$

and for m of the one-pole-one-zero high-pass filters we obtain solution

$$\alpha = \frac{2^{1/m} - 2\left(\sqrt{2^{1/m} - 1}\right)\sin(\omega_c) + \left(2 - 2^{1/m}\right)\cos(\omega_c)}{2^{1/m}\cos(\omega_c) + \left(2 - 2^{1/m}\right)}.$$

Further Reading

"Music: A Mathematical Offering" - David J. Benson, 2007, Cambridge University Press
https://ccrma.stanford.edu/~jos/filters/index.html
http://freeverb3.sourceforge.net/doc/AN11.pdf
http://inst.eecs.berkeley.edu/~ee247/fa05/lectures/L2_f05.pdf
http://www-sigproc.eng.cam.ac.uk/~op205/3F3_4_Basics_of_Digital_FIlters.pdf