First-Order Digital Filters

(version 0.2)

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Abstract

Linear time-invariant digital recursive filters are often constructed by applying a transformation to an equivalent analogue filter. This involves mapping points in the Laplace transform s-plane to points in the Z-transform z-plane through such means as the bilinear transformation or the impulse-invariance method. However, these digital filters can of course be constructed directly in the z-plane. The present work is an analysis of this direct approach for first-order filters, requiring only simple calculus and algebraic techniques.

(This document is a work in progress.)

Chapter 0

Zeroth-Order

The simplest possible filter is of zeroth order. Consider the equation

$$a_0 y_n = b_0 x_n \tag{0.1}$$

where x_n, y_n are sample data and $n \in \mathbb{Z}$. If $a_0 = 0$, then $b_0 = 0$ since $x_n \neq 0$ in general. So with $a_0 \neq 0$ we can write (0.1) in the form

$$y_n = \gamma x_n \tag{0.2}$$

where γ is a constant. Applying the Z-transform to (0.2) yields the *transfer function* $H(z)$:

$$H(z) = \frac{Y(z)}{X(z)} \tag{0.3}$$

$$= \gamma. \tag{0.4}$$

The *frequency response* of the filter is obtained by evaluating the transfer function $H(z)$ along the unit circle $z = \mathrm{e}^{\imath\omega}$ for all ω, where

$$\omega = \frac{2\pi f}{f_s}, \tag{0.5}$$

f is the input frequency and f_s is the *sampling rate*. The *amplitude response* of the filter is provided by the *gain* $G(\omega)$:

$$G(\omega) = |H(\mathrm{e}^{\imath\omega})| \tag{0.6}$$

$$= |\gamma|, \tag{0.7}$$

similarly, the *phase response* of the filter is provided by the *phase* $\Theta(\omega)$:

$$\Theta(\omega) = \arg\left(H(\mathrm{e}^{\imath\omega})\right) \tag{0.8}$$

$$= \arg(\gamma). \tag{0.9}$$

Since $H(\mathrm{e}^{\imath\omega})$ is periodic with period 2π, both the gain $G(\omega)$ and phase $\Theta(\omega)$ are also periodic with period 2π. In fact for this zeroth-order case the gain and phase are constants and so are periodic with respect to any period.

To obtain real output data y_n from real input data x_n, we require $\gamma = r\mathrm{e}^{\imath\theta}$ to be real. If $\gamma > 0$, then $\theta = 2n\pi, n \in \mathbb{Z}$. Since Θ is 2π periodic, we may take $n = 0$. Thus $\gamma > 0$ adjusts the overall input gain with no phase shift. If $\gamma < 0$, then $\theta = (2n+1)\pi, n \in \mathbb{Z}$. Taking $n = 0$ we see that $\gamma < 0$ adjusts the overall input gain and applies a phase shift of π, equivalent to inverting the input. If $\gamma = 0$, then $G(\omega) = 0$ for all ω and the output is always zero. In this case θ can assume any value. Thus (0.9) becomes

$$\Theta(\omega) = \begin{cases} 0, & \gamma > 0 \\ \text{undefined}, & \gamma = 0 \\ \pi, & \gamma < 0. \end{cases} \tag{0.10}$$

For the zeroth order filter, the gain or phase at any particular frequency is no more or less than the gain or phase at any other frequency.

Chapter 1

First-Order

Consider the linear first-order *difference equation* with constant coefficients

$$a_0 y_n + a_1 y_{n-1} = b_0 x_n + b_1 x_{n-1}. \tag{1.1}$$

If $a_0 = 0$ and $b_0 = 0$, then the first-order filter reduces to the zeroth-order case. If $a_0 = 0$ and $a_1 = 0$, then $b_0 = 0$ and $b_1 = 0$, since b_0 and b_1 cannot depend upon the input data. If $a_0 = 0$ and $b_0 \neq 0$, then the output y_{n-1} depends upon the 'future' input x_n (and possibly the 'current' input x_{n-1}). Such a filter is said to be *acausal*. We will concern ourselves only with *causal* filters, containing no future inputs or outputs. Thus we assume $a_0 \neq 0$, allowing us to write (1.1) in the equivalent form

$$y_n = b_0 x_n + b_1 x_{n-1} - a_1 y_{n-1}. \tag{1.2}$$

Applying the Z-transform to (1.2) yields the transfer function

$$\begin{aligned} H(z) &= Y(z)/X(z) && (1.3)\\ &= \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} && (1.4)\\ &= \gamma \frac{(1 - \beta z^{-1})}{(1 - \alpha z^{-1})} && (1.5) \end{aligned}$$

where

$$b_0 = \gamma,\ b_1 = -\beta\gamma,\ a_1 = -\alpha. \tag{1.6}$$

α is called the *pole* of the filter, giving the z location where the denominator of (1.5) is zero. β is called the *zero* of the filter, giving the location where the numerator of (1.5) is zero. To obtain real output data y_n from real input data x_n, we require the coefficients in (1.2) to be real, hence α, β and γ must also be real by (1.6). If $|\alpha| > 1$, then $|a_1| > 1$, yielding an *unstable* filter. If $|\alpha| = 1$ the filter is said to be *marginally stable*; if $|\alpha| < 1$ the filter is said to be *stable*. There is no such stability restriction on β or γ.

The gain of the filter is

$$\begin{aligned} G(\omega) &= |H(e^{\imath\omega})| && (1.7)\\ &= |\gamma| \sqrt{\frac{1 - 2\beta\cos\omega + \beta^2}{1 - 2\alpha\cos\omega + \alpha^2}} && (1.8) \end{aligned}$$

and the phase

$$\begin{aligned} \Theta(\omega) &= \arg\left(H(e^{\imath\omega})\right) && (1.9)\\ &= \arg(\gamma) + \tan^{-1}\left\{\frac{\beta\sin\omega}{1 - \beta\cos\omega}\right\} - \tan^{-1}\left\{\frac{\alpha\sin\omega}{1 - \alpha\cos\omega}\right\} && (1.10)\\ &\equiv \arg(\gamma) + \tan^{-1}\left\{\frac{(\beta - \alpha)\sin\omega}{1 + \alpha\beta - (\alpha + \beta)\cos\omega}\right\}. && (1.11) \end{aligned}$$

The derivative of (1.8) is

$$\frac{dG}{d\omega} = \frac{|\gamma|(\alpha - \beta)(\alpha\beta - 1)\sin\omega}{\sqrt{(1 - 2\beta\cos\omega + \beta^2)(1 - 2\alpha\cos\omega + \alpha^2)^3}} \tag{1.12}$$

and of (1.10) is

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\omega} = \frac{\beta(\cos\omega - \beta)}{1 - 2\beta\cos\omega + \beta^2} - \frac{\alpha(\cos\omega - \alpha)}{1 - 2\alpha\cos\omega + \alpha^2} \tag{1.13}$$

$$\equiv \frac{(\alpha - \beta)\big((\alpha + \beta) - (1 + \alpha\beta)\cos\omega\big)}{(1 + \alpha^2)(1 + \beta^2) - 2(\alpha + \beta)(1 + \alpha\beta)\cos\omega + 4\alpha\beta\cos^2\omega}. \tag{1.14}$$

Observations

Reciprocal Solutions
Substitute $\beta_{\mathrm{rcp}} = 1/\beta$ into (1.8):

$$G(\omega) = \frac{|\gamma|}{|\beta_{\mathrm{rcp}}|}\sqrt{\frac{1 - 2\beta_{\mathrm{rcp}}\cos\omega + \beta_{\mathrm{rcp}}^2}{1 - 2\alpha\cos\omega + \alpha^2}}. \tag{1.15}$$

If $|\gamma| = |\gamma_\beta|$ for $|\beta| < 1$ and $|\gamma| = |\beta_{\mathrm{rcp}}||\gamma_\beta|$ for $|\beta| \geqslant 1$, then (1.8) and (1.15) yields the same equation for β and its reciprocal β_{rcp} at any chosen frequency. (A similar argument applies to α however we can only choose one $|\alpha| < 1$ for stability.)

Zero Gain
Provided the denominator of (1.8) is non-zero, the gain is trivially zero when $|\gamma| = 0$ and more interestingly zero when

$$\beta^2 - 2\beta\cos\omega_{0_G} + 1 = 0 \tag{1.16}$$

for some frequencies ω_{0_G}. (1.16) has solution

$$\beta = \cos\omega_{0_G} \pm \sqrt{\cos\omega_{0_G}^2 - 1} \tag{1.17}$$

which is real only when $\cos\omega_{0_G} = \pm 1$, i.e. when $\omega_{0_G} = 2n\pi, n \in \mathbb{Z}$, the *DC* frequency, or when $\omega_{0_G} = (2n+1)\pi, n \in \mathbb{Z}$, the *Nyquist* frequency. At these frequencies $\beta = 1$ and $\beta = -1$ respectively.

Infinite Gain
Provided the numerator of (1.8) is non-zero, the gain is infinite when

$$\alpha^2 - 2\alpha\cos\omega_{\infty_G} + 1 = 0 \tag{1.18}$$

for some frequencies ω_{∞_G}. (1.18) is the same form as (1.16) and has real solutions for α only at DC and Nyquist, i.e. when $\alpha = 1$ and $\alpha = -1$ respectively.

Average Gain
Let the gain at DC be G_{DC} and that at Nyquist G_{Nyq}. Consider the geometric mean

$$G_{\mathrm{avg}} = \sqrt{G_{\mathrm{DC}} G_{\mathrm{Nyq}}} \tag{1.19}$$

$$= |\gamma|\sqrt{\frac{|1 - \beta^2|}{1 - \alpha^2}} \tag{1.20}$$

by evaluating (1.8) at DC and Nyquist in turn. We also have that

$$G_{\mathrm{avg}} = |\gamma|\sqrt{\frac{1 - 2\beta\cos\omega_{\mathrm{avg}_G} + \beta^2}{1 - 2\alpha\cos\omega_{\mathrm{avg}_G} + \alpha^2}} \tag{1.21}$$

by evaluating (1.8) at $G_{\mathrm{avg}} = G(\omega_{\mathrm{avg}_G})$. Suppose $|\beta| \leqslant 1$. Then eliminating $\frac{G_{\mathrm{avg}}}{|\gamma|}$ from (1.20) and (1.21) we find

$$(\beta - \alpha)\big((1 - \alpha\cos\omega_{\mathrm{avg}_G})\beta + \alpha - \cos\omega_{\mathrm{avg}_G}\big) = 0. \tag{1.22}$$

If $\beta = \alpha$, the gain is constant, and ω_{avg_G} can assume any value. If $\beta \neq \alpha$, then

$$\cos\omega_{\mathrm{avg}_G} = \frac{\alpha + \beta}{1 + \alpha\beta}, \quad |\beta| \leqslant 1. \tag{1.23}$$

Now suppose $|\beta| \geqslant 1$. Eliminating $\frac{G_{\text{avg}}}{|\gamma|}$ from (1.20) and (1.21) we find

$$(1-\alpha\beta)\big((\alpha-\cos\omega_{\text{avg}_G})\beta+1-\alpha\cos\omega_{\text{avg}_G}\big)=0. \tag{1.24}$$

If $\beta = 1/\alpha$, the gain is again constant, and ω_{avg_G} can assume any value. If $\beta \neq 1/\alpha$, then

$$\cos\omega_{\text{avg}_G}=\frac{1+\alpha\beta}{\alpha+\beta}, \quad |\beta|\geqslant 1 \tag{1.25}$$

which is (1.23) with β replaced by $1/\beta$. We expect this result due to the reciprocal nature of β.

Cutoff Gain
Suppose we are given a frequency ω_{ref} at which the gain is G_{ref} and we want to find a frequency ω_{kc} such that

$$\begin{aligned} G^2(\omega_{\text{kc}}) &= \kappa G_{\text{ref}}^2 \\ &= G_{\text{kc}}^2 \end{aligned} \tag{1.26}$$

with $0 < \kappa < 1$. We call ω_{kc} a *cutoff* frequency. The cutoff frequency denotes a transition from a *passband* (very little attenuation of input signal frequencies) to a *stopband* (large attenuation of input signal frequencies). It follows from (1.8) that

$$\big((1-\kappa)(1+\alpha^2)-2(\cos\omega_{\text{ref}}-\kappa\cos\omega_{\text{kc}})\alpha\big)(\beta^2+1)-2\big((\cos\omega_{\text{kc}}-\kappa\cos\omega_{\text{ref}})(1+\alpha^2)-2(1-\kappa)(\cos\omega_{\text{ref}}\cos\omega_{\text{kc}})\alpha\big)\beta=0. \tag{1.27}$$

If, in addition, we set $\kappa = 1/2$, we have that

$$\big((1+\alpha^2)+2\alpha(\cos\omega_{\text{hc}}-2\cos\omega_{\text{ref}})\big)(\beta^2+1)+2\big((\cos\omega_{\text{ref}}-2\cos\omega_{\text{hc}})(1+\alpha^2)+2\alpha\cos\omega_{\text{ref}}\cos\omega_{\text{hc}}\big)\beta=0 \tag{1.28}$$

where we have substituted ω_{hc} for ω_{kc}.

Zero Gain Derivative
Provided the denominator of (1.12) is non-zero, the derivative is trivially zero when $\gamma = 0$ (for which the gain is zero), or when $\beta = \alpha$ or $\beta = 1/\alpha$ (for which the gain is constant). The derivative is more interestingly zero when

$$\sin\omega_{0_{\text{d}G}}=0 \tag{1.29}$$

for frequencies $\omega_{0_{\text{d}G}}$, i.e. at DC or Nyquist. It follows that if the gain is not constant it is strictly monotonic between DC and Nyquist.

Infinite Gain Derivative
Provided the numerator of (1.12) is non-zero, the derivative is only infinite in the uninteresting cases: DC frequency and $\beta = 1$ or $\alpha = 1$; Nyquist frequency and $\beta = -1$ or $\alpha = -1$.

Quadrature Phase
Due to the periodic nature of Θ it is the phase difference between two frequencies rather than their absolute values that is important. Let the phase at DC be $\Theta_{\text{DC}} = 0$. The filter output frequency is said to be in *quadrature* with the filter input frequency when $\Theta = \pi/2$. Using (1.10), this occurs for frequency(ies) $\cos\omega_{\pi/2}$ satisfying

$$\cos\omega_{\pi/2}=\frac{1+\alpha\beta}{\alpha+\beta}. \tag{1.30}$$

Zero Phase Derivative
Provided the denominator of (1.13) is non-zero, the derivative is trivially zero when $\beta = \alpha$ or more interestingly zero when

$$\cos\omega_{0_{\text{d}\Theta}}=\frac{\alpha+\beta}{1+\alpha\beta} \tag{1.31}$$

which gives the frequencies $\omega_{0_{\text{d}\Theta}}$ of maximum phase deviation.

Infinite Phase Derivative
Provided the relevant numerator of (1.13) is non-zero, the derivative is only infinite in the uninteresting cases: DC frequency and $\beta = 1$ or $\alpha = 1$; Nyquist frequency and $\beta = -1$ or $\alpha = -1$.

Average Phase
Consider the arithmetic mean

$$\Theta_{\text{avg}} = \frac{\Theta_{\text{DC}} + \Theta_{\text{Nyq}}}{2} \tag{1.32}$$

occurring at frequencies $\omega_{\text{avg}_\Theta}$. If (1.31) is not satisfied, then the phase is strictly monotonic between $0 \leqslant \omega \leqslant \pi$. Since $\Theta_{\text{Nyq}} - \Theta_{\text{DC}} = n\pi$, $n \in \mathbb{Z}$, it follows that $n \neq 0$, and since there is only one real $0 \leqslant \omega_{\pi/2} \leqslant \pi$ satisfying (1.30), we must have that $n = 1$. Thus if we set $\Theta_{\text{DC}} = 0$ then $\Theta_{\text{Nyq}} = \pi$ and $\Theta_{\text{avg}} = \pi/2$. Alternatively, if (1.31) is satisfied, then $\Theta_{\text{DC}} = 0$ implies $\Theta_{\text{Nyq}} = 0$, and the maximum phase deviation from either DC or Nyquist is $\pi/2$.

Gain & Phase Symmetry
By inspection of (1.23), (1.25), (1.30) and (1.31) we conclude that

$$\omega_{\text{avg}_G} = \omega_{0_{\text{d}\Theta}}, \quad |\beta| \leqslant 1 \tag{1.33}$$

$$= \omega_{\text{avg}_\Theta}, \quad |\beta| \geqslant 1. \tag{1.34}$$

We will now look at some of the first-order filter types that can be designed.

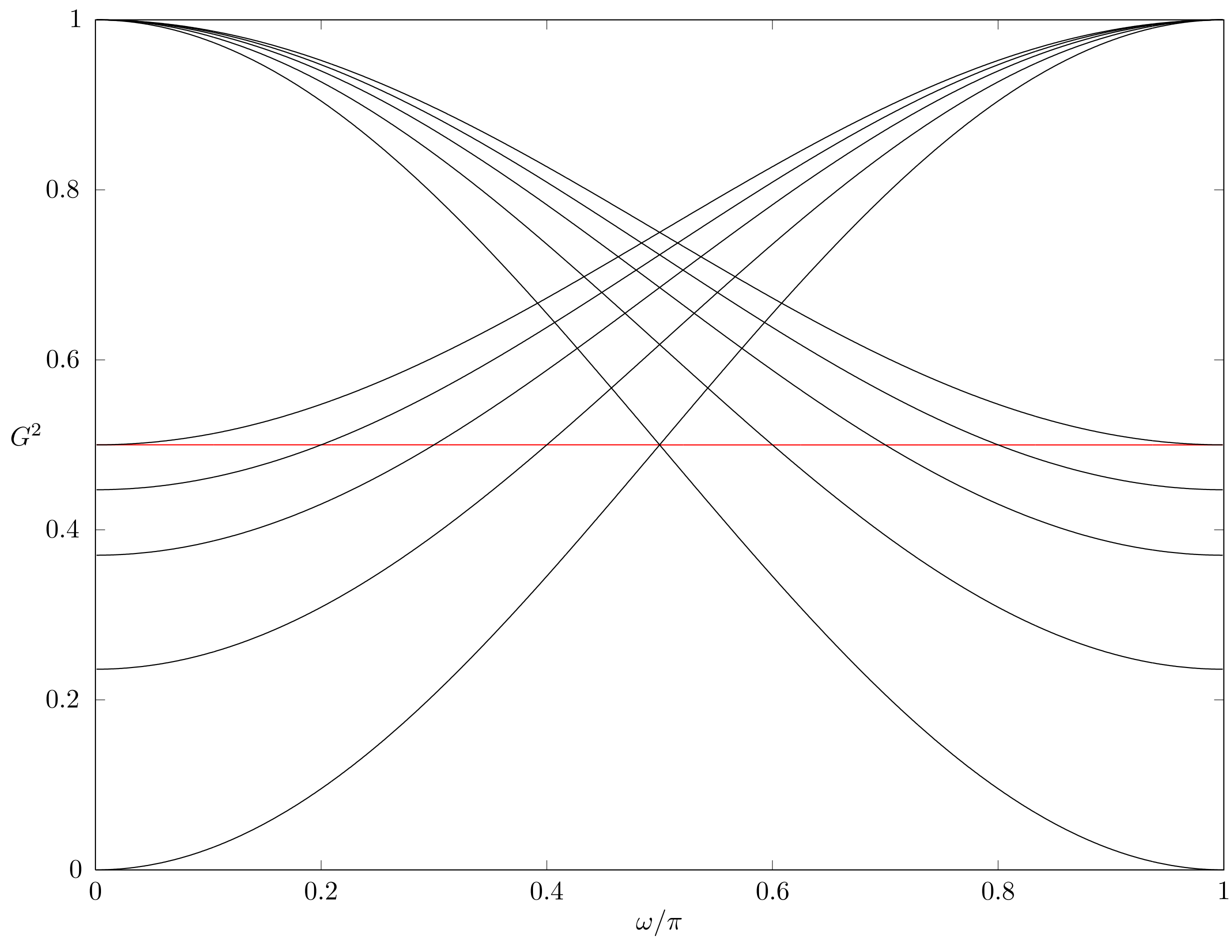

Figure 1.1: One-zero filter gain: High-pass, $\omega_{\text{hc}}/\pi = 0$, 0.2, 0.3, 0.4, 0.5;
Low-pass, $\omega_{\text{hc}}/\pi = 0.5$, 0.6, 0.7, 0.8, 1

1.1 One-zero

Let's examine the case $\alpha = 0$. Then the filter has a single zero, and the output only depends on the current and the previous input. Filters whose output depends only upon current, previous and/or future inputs are called *Finite Impulse Response (FIR)* filters, or sometimes *convolution* filters. With a single zero, the gain (1.8) simplifies to

$$G(\omega) = |\gamma|\sqrt{1 - 2\beta\cos\omega + \beta^2} \tag{1.35}$$

and the phase (1.10) simplifies to

$$\Theta(\omega) = \arg(\gamma) + \tan^{-1}\left\{\frac{\beta\sin\omega}{1-\beta\cos\omega}\right\}. \tag{1.36}$$

We can construct two types of one-zero filter, a *low-pass* filter, letting low frequencies through whilst blocking high frequencies, and a *high-pass* filter, which lets high frequencies through whilst blocking low frequencies.

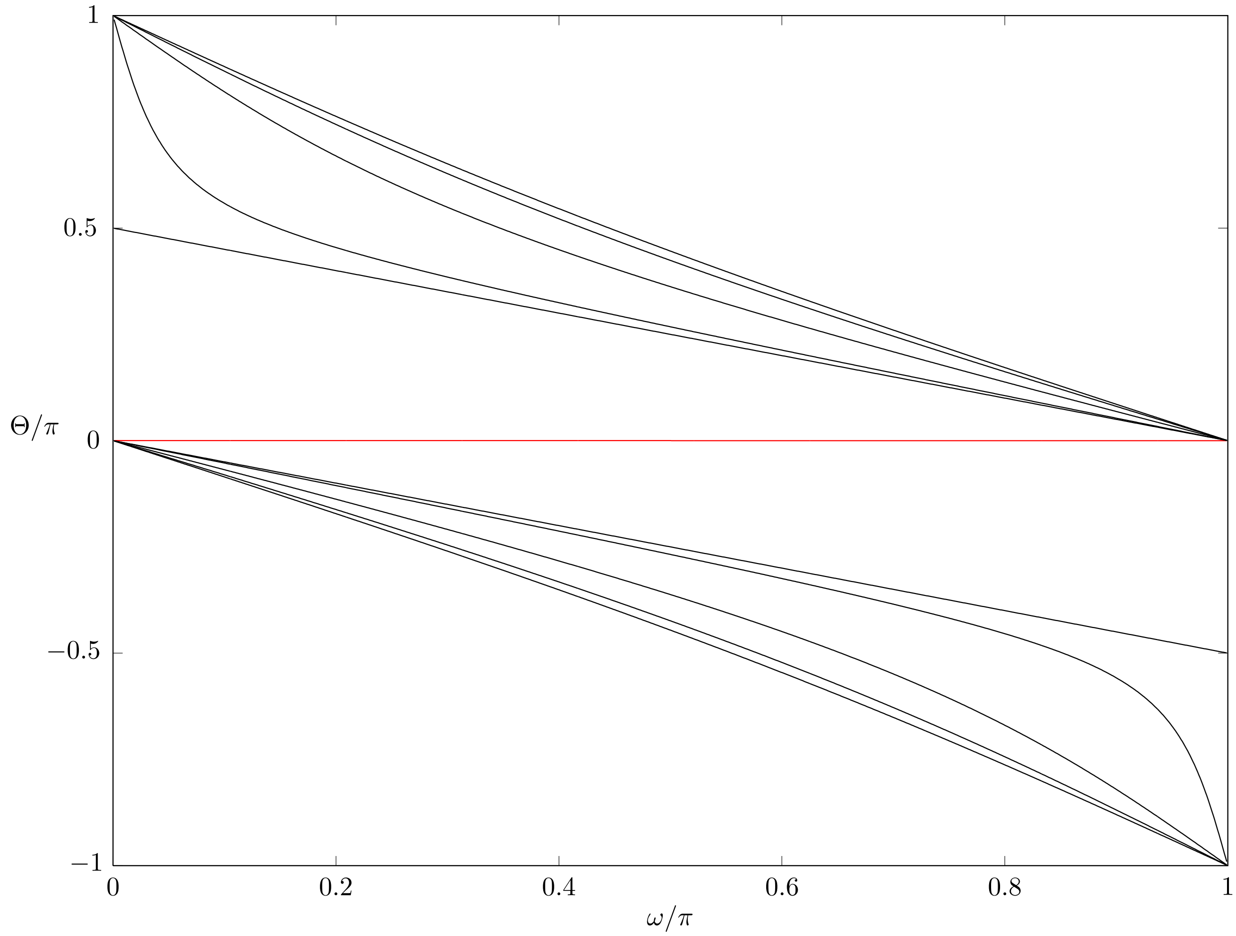

Figure 1.2: One-zero filter phase, $|\beta| > 1$: High-pass (top), $\omega_{\text{hc}}/\pi = 0.5,\ 0.499,\ 0.45,\ 0.3,\ 0$; Low-pass (bottom), $\omega_{\text{hc}}/\pi = 1,\ 0.7,\ 0.55,\ 0.501,\ 0.5$

1.1.1 Low-pass

If $\beta < 0$, we see from (1.35) that frequencies near Nyquist will be attenuated whilst those near DC will be boosted. Let's use γ to normalise the gain at DC to unity, i.e. set $G(0) = 1$. We obtain

$$\gamma = \frac{1}{|1-\beta|}, \tag{1.37}$$

the choice of sign of γ to be explained shortly. Now let $\omega = \omega_{\text{hc}}$ be a cutoff frequency where

$$G_{\text{hc}}^2 = 1/2 \tag{1.38}$$

with $G_{\text{hc}} = G(\omega_{\text{hc}})$. Here the cutoff denotes the frequency at which the power ($\propto G^2$) is half of the maximum; this is a very common definition. In *decibel* notation $1/2$ becomes $10\log_{10}(1/2)\,\text{dB} \approx -3.0103\,\text{dB}$ which is often stated approximately as $-3\,\text{dB}$. From (1.28) β must satisfy

$$\beta^2 + 2(1-2\cos\omega_{\text{hc}})\beta + 1 = 0, \tag{1.39}$$

which is a quadratic with solution

$$\beta = -1 + 2\cos\omega_{\text{hc}} - 2\sqrt{(\cos\omega_{\text{hc}} - 1)\cos\omega_{\text{hc}}} \tag{1.40}$$

where we have chosen the root which keeps the zero outside the unit circle. One observation of (1.40) is that $\pi/2 \leqslant \omega_{\text{hc}} \leqslant \pi$ in order for β to stay real. This is a major restriction of our one-zero low-pass filter. A second observation is that if ω_{hc} is close to $\pi/2$, frequencies near Nyquist are attenuated very strongly. In fact, if $\omega_{\text{hc}} = \pi/2$ we have $\beta = -1$ and frequencies at Nyquist are completely blocked. This is a major benefit of our one-zero low-pass filter.

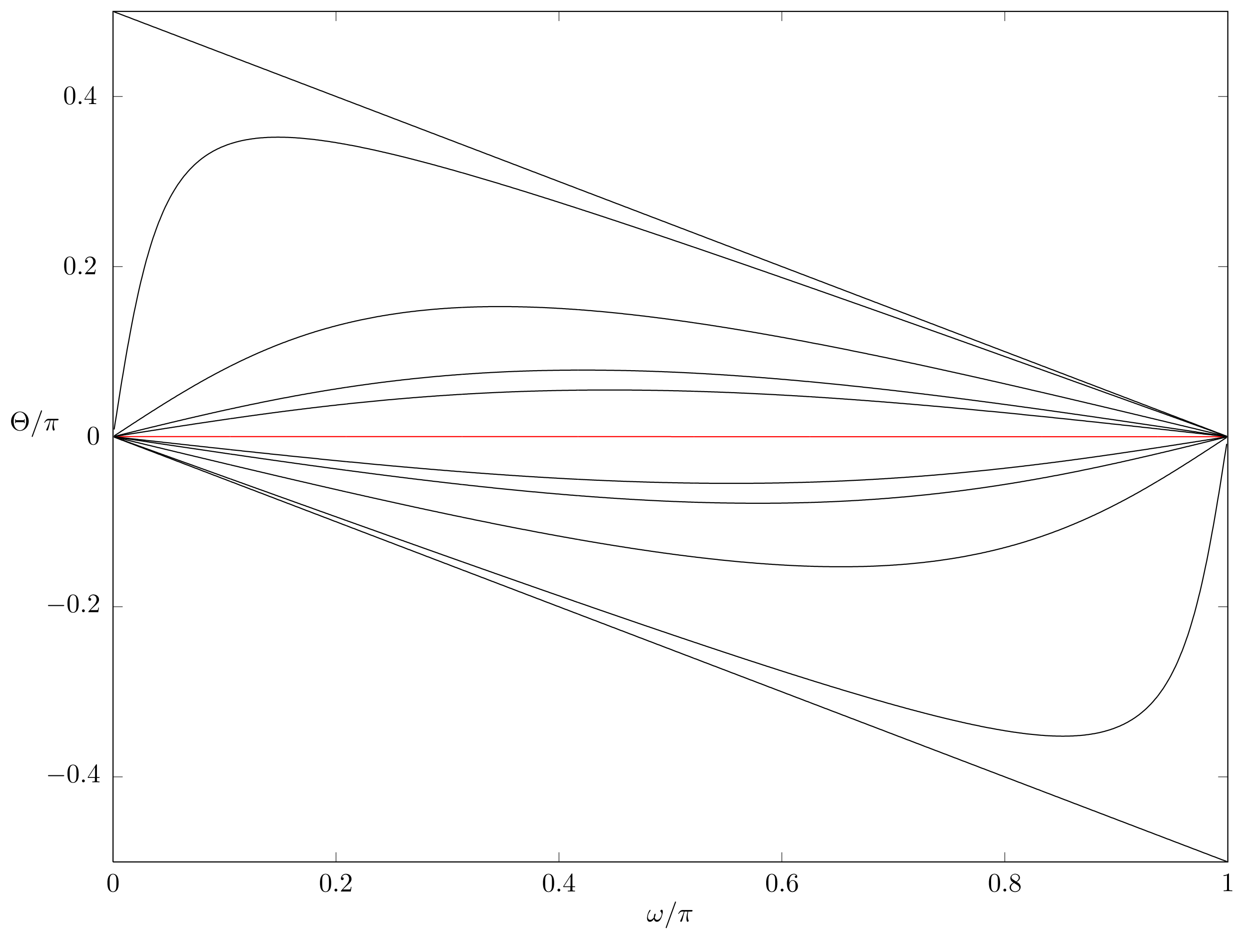

Figure 1.3: One-zero filter phase, $|\beta| < 1$: High-pass (top), $\omega_{\text{hc}}/\pi = 0.5$, 0.499, 0.45, 0.3, 0; Low-pass (bottom), $\omega_{\text{hc}}/\pi = 0.5$, 0.501, 0.55, 0.7, 1

1.1.2 High-pass

If $\beta > 0$, we see from (1.35) that frequencies near DC will be attenuated whilst those near Nyquist will be boosted. Normalising the gain to unity at Nyquist, i.e. setting $G(\pi) = 1$, we obtain

$$\gamma = \frac{1}{|1+\beta|}, \tag{1.41}$$

the choice of sign of γ to be explained shortly. Choose β such that $G_{\text{hc}}^2 = 1/2$:

$$\beta^2 - 2(1 + 2\cos\omega_{\text{hc}})\beta + 1 = 0 \tag{1.42}$$

which has solution

$$\beta = 1 + 2\cos\omega_{\text{hc}} + 2\sqrt{(1 + \cos\omega_{\text{hc}})\cos\omega_{\text{hc}}} \tag{1.43}$$

choosing the root which keeps the zero outside the unit circle. Here (1.43) requires that $0 \leqslant \omega_{\text{hc}} \leqslant \pi/2$ in order for β to stay real, a major restriction of our one-zero high-pass filter. When ω_{hc} is close to $\pi/2$, near

DC frequencies are attenuated very strongly. In fact, if $\omega_{hc} = \pi/2$ we have $\beta = 1$ and frequencies at DC are completely blocked. This is a major benefit of our one-zero high-pass filter. We notice that the low-pass and high-pass one-zero filters are essentially identical in character.

Choice of roots
For the one-zero low-pass and high-pass solutions considered above, if $|\beta| > 1$, i.e. if β lies outside the unit circle, then there are no real frequencies where the phase has a local maximum or minimum. The overall phase response may be preferred in this case, being of an approximately linear nature. Alternatively, if $|\beta| < 1$, then the phase deviation is minimised. Such a filter is called a *minimum phase* filter. A filter with a completely linear phase response is called a *linear phase* filter. Only FIR filters can achieve linear phase.

Choice of sign of γ
The choice is such that in the low-pass case, frequencies near DC are phase-shifted as little as possible; in the high-pass case, frequencies near Nyquist are phase-shifted as little as possible. This is achieved by setting $\gamma > 0$ in both cases.

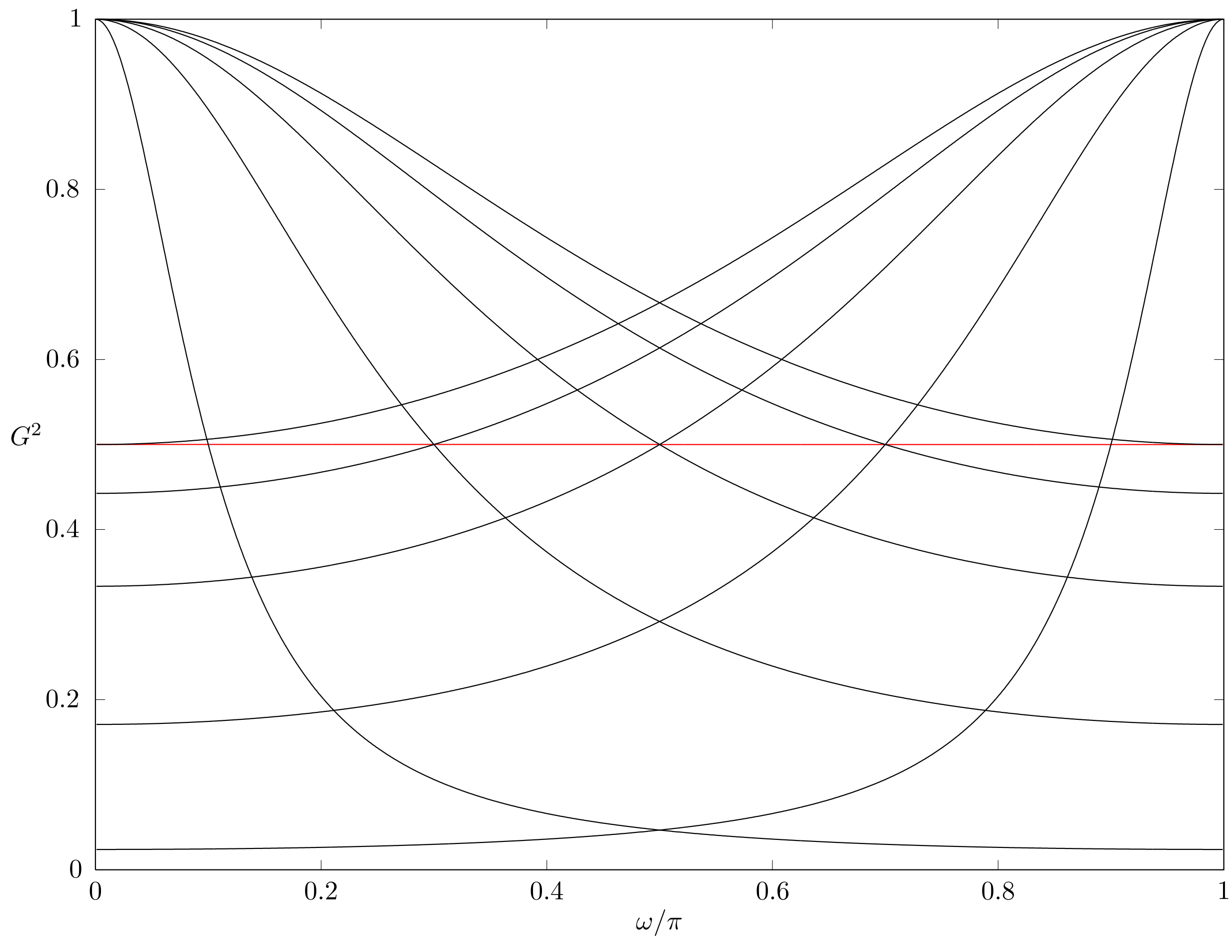

Figure 1.4: One-pole filter gain: High-pass, $\omega_{hc}/\pi = 0$, 0.3, 0.5, 0.7, 0.9;
Low-pass, $\omega_{hc}/\pi = 0.1$, 0.3, 0.5, 0.7, 1

1.2 One-pole

If $\beta = 0$ we have a single pole, with the output depending upon the current input and the previous output. Filters whose output depends on previous and/or future outputs, whether or not the output also depends on current, previous and/or future inputs, are called *Infinite Impulse Response (IIR)* filters, or *recursive* filters. They are 'infinite' in the sense that once an input is fed into the filter, the output can never completely decay to zero. (Aside: When such filters are coded, care must be taken to ensure that the floating point unit of the target processor is able to deal with the small numbers that inevitably arise - this is often referred to as the ‘denormal’ problem.) The gain (1.8) simplifies to

$$G(\omega) = \frac{|\gamma|}{\sqrt{1 - 2\alpha\cos\omega + \alpha^2}} \tag{1.44}$$

and the phase (1.10) simplifies to

$$\Theta(\omega) = \arg(\gamma) - \tan^{-1}\left\{\frac{\alpha\sin\omega}{1 - \alpha\cos\omega}\right\}. \tag{1.45}$$

We can construct two types of one-pole filter, a low-pass and a high-pass.

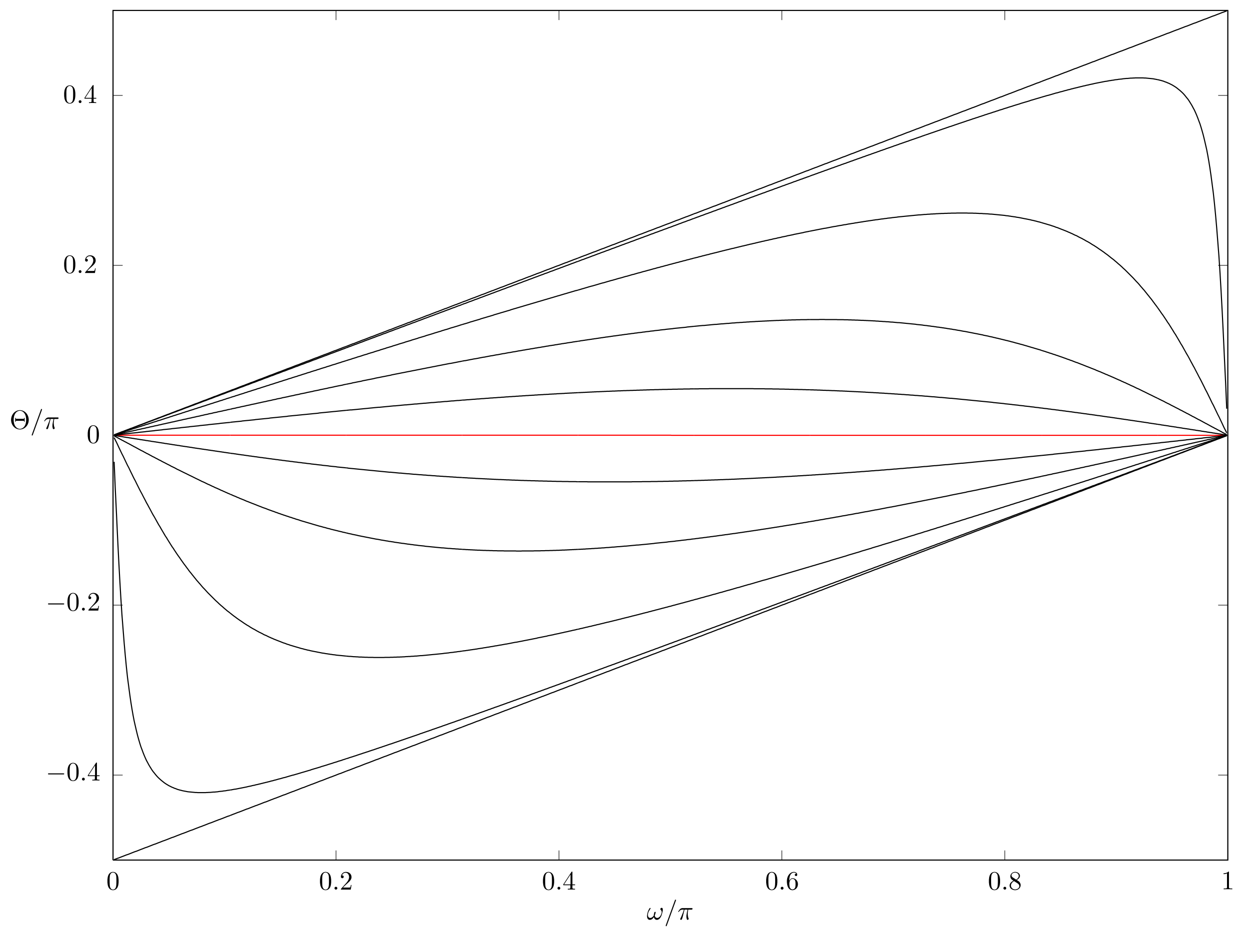

Figure 1.5: One-pole filter phase: High-pass (top), $\omega_{\rm hc}/\pi = 0,\ 0.7,\ 0.9,\ 0.99,\ 1$;
Low-pass (bottom), $\omega_{\rm hc}/\pi = 0,\ 0.01,\ 0.1,\ 0.3,\ 1$

1.2.1 Low-pass

If $\alpha > 0$, we see from (1.44) that frequencies near DC will be boosted whilst those near Nyquist will be attenuated. Normalising the gain at DC to unity:

$$\gamma = 1 - \alpha, \tag{1.46}$$

where we have set $\gamma > 0$. Notice it is not necessary to take the modulus $|1 - \alpha|$ as we similarly did in the one-zero case since we require $|\alpha| < 1$ for stability. Choose α such that $G^2_{\rm hc} = {}^1\!/_2$:

$$\alpha^2 - 2(2 - \cos\omega_{\rm hc})\alpha + 1 = 0 \tag{1.47}$$

which has solution

$$\alpha = 2 - \cos\omega_{\rm hc} - \sqrt{(\cos\omega_{\rm hc} - 3)(\cos\omega_{\rm hc} - 1)}. \tag{1.48}$$

where we have chosen the appropriate root so that $|\alpha| \leqslant 1$. Unlike the one-zero low-pass case, there is no restriction on $\omega_{\rm hc}$; α is always real. On the other hand, the one-zero filter can perform much better near Nyquist. The one-pole low-pass filter is the most commonly used first-order low-pass filter algorithm.

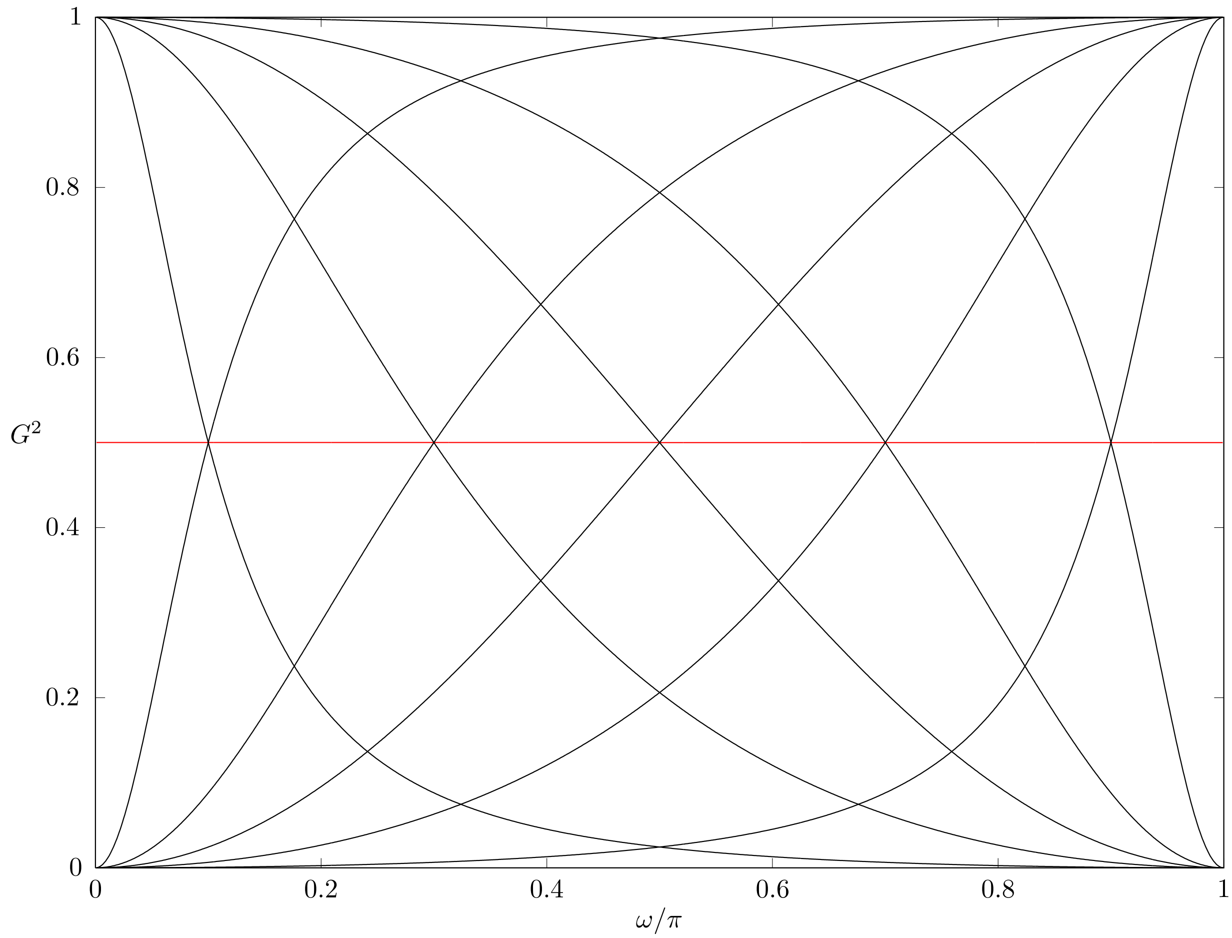

Figure 1.6: One-pole one-zero filter gain: High-pass, $\omega_{\text{hc}}/\pi = 0.1, 0.3, 0.5, 0.7, 0.9$;
Low-pass, $\omega_{\text{hc}}/\pi = 0.1, 0.3, 0.5, 0.7, 0.9$

1.2.2 High-pass

If $\alpha < 0$, we see from (1.44) that frequencies near Nyquist will be boosted whilst those near DC will be attenuated. Normalising the gain at Nyquist to unity:

$$\gamma = 1 + \alpha, \tag{1.49}$$

setting $\gamma > 0$. Choose α such that $G^2_{\text{hc}} = 1/2$:

$$\alpha^2 + 2(2 + \cos\omega_{\text{hc}})\alpha + 1 = 0 \tag{1.50}$$

which has solution

$$\alpha = -2 - \cos\omega_{\text{hc}} + \sqrt{(\cos\omega_{\text{hc}} + 3)(\cos\omega_{\text{hc}} + 1)}. \tag{1.51}$$

where we have chosen the appropriate root so that $|\alpha| \leqslant 1$. Unlike the one-zero high-pass case and as for the one-pole low-pass case, there is no restriction on ω_{hc}; α is always real. However, the one-zero high-pass filter can perform much better near DC.

1.3 One-pole One-zero

The best-performing first-order low-pass and high-pass filters are of the one-pole one-zero type (as one might expect). We can also construct other filter types.

1.3.1 Low-pass

To improve upon the one-pole low-pass filter, let's place the zero at Nyquist – the most natural choice. Thus set

$$\beta = -1 \tag{1.52}$$

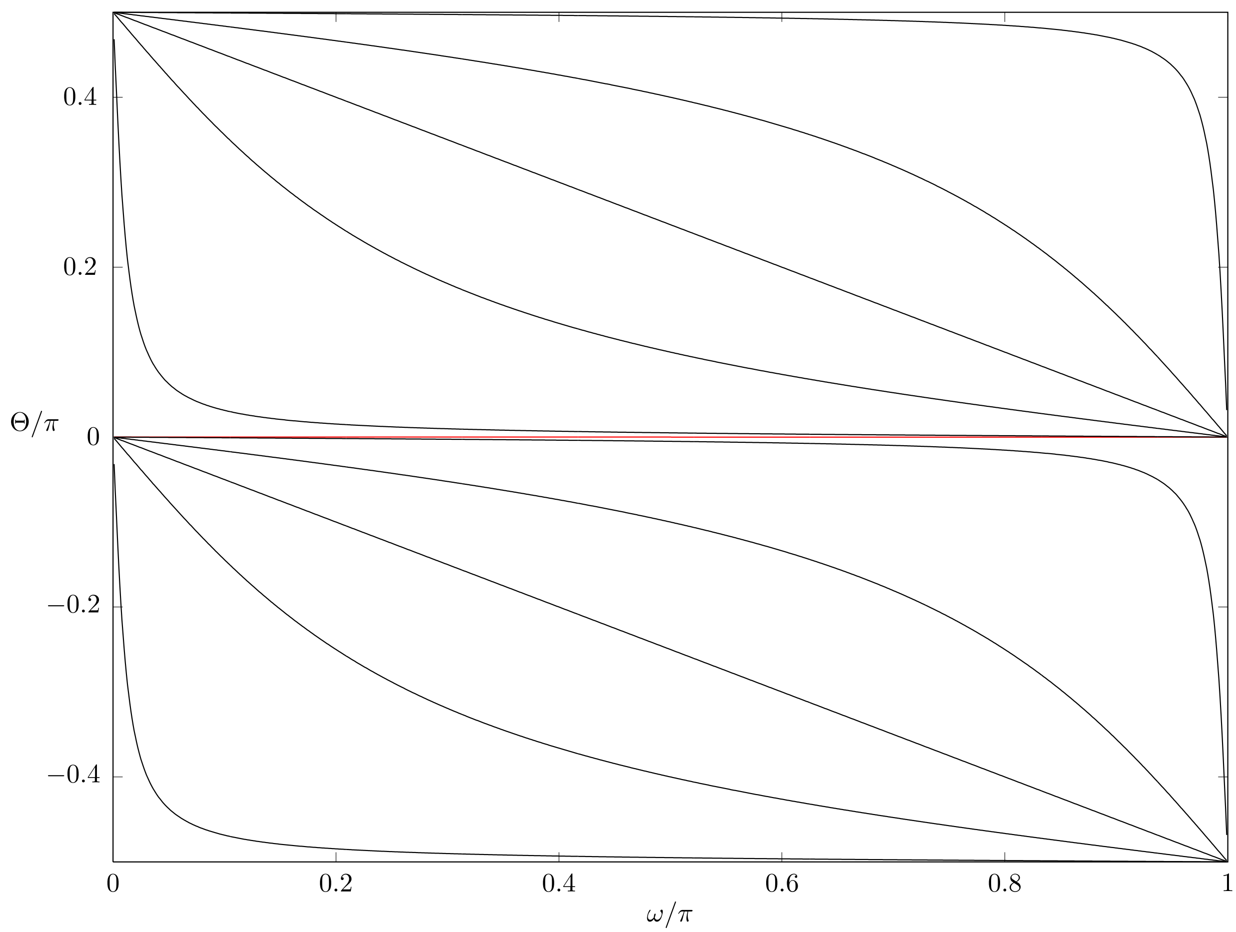

Figure 1.7: One-pole one-zero filter phase: High-pass (top), $\omega_{\text{hc}}/\pi = 0.01$, 0.2, 0.5, 0.8, 0.99; Low-pass (bottom), $\omega_{\text{hc}}/\pi = 0.01$, 0.2, 0.5, 0.8, 0.99

then normalise the gain at DC to unity

$$\gamma = \frac{1-\alpha}{2} \tag{1.53}$$

then set $G_{\text{hc}}^2 = 1/2$

$$(\alpha^2 + 1)\cos\omega_{\text{hc}} - 2\alpha = 0 \tag{1.54}$$

which has solution

$$\alpha = \frac{1 - \sin\omega_{\text{hc}}}{\cos\omega_{\text{hc}}} \tag{1.55}$$

selecting the appropriate root so that α is finite at $\omega_{\text{hc}} = \pi/2$. As for the one-pole low-pass case there are no restrictions on ω_{hc}, although care must be taken if coding this filter when $\omega_{\text{hc}} \approx \pi/2$. This filter performs better than the one-pole low-pass filter.

1.3.2 High-pass

To improve upon the one-pole high-pass filter, let's place the zero at DC – the most natural choice. Thus set

$$\beta = 1 \tag{1.56}$$

then normalise the gain at Nyquist to unity

$$\gamma = \frac{1+\alpha}{2} \tag{1.57}$$

then set $G_{\text{hc}}^2 = 1/2$, which yields the same equation (1.54) and solution (1.55) for α as found in the low-pass case. So again there are no restrictions on ω_{hc}. This filter performs (much) better than the one-pole high-pass filter. The one-pole one-zero high-pass filter is the most commonly used first-order high-pass filter algorithm, where it is often (mistakenly) labelled as only one-pole.

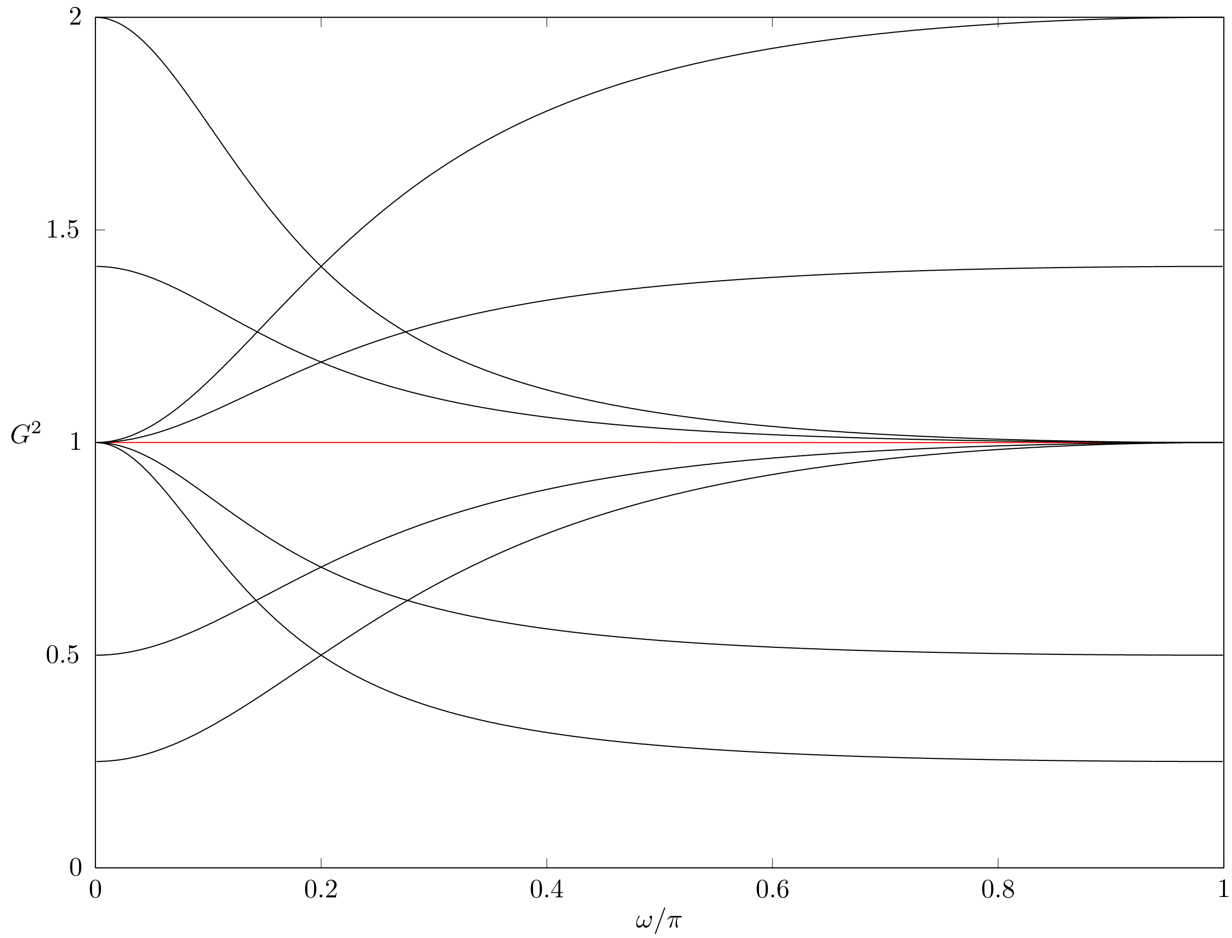

Figure 1.8: First-order shelf gain, $\omega_{\text{avg}}/\pi = 0.2$: Low-shelf, $G_{\text{DC}} = 0.5$, $\sqrt{0.5}$, $\sqrt[4]{2}$, $\sqrt{2}$; High-shelf, $G_{\text{Nyq}} = 0.5$, $\sqrt{0.5}$, $\sqrt[4]{2}$, $\sqrt{2}$

1.3.3 Shelving

Instead of specifying the gain to be zero at Nyquist or DC, we can specify any value. This yields a type of filter called a *shelving* filter. Suppose $G(\pi) = G_{\text{Nyq}}$ and $G(0) = G_{\text{DC}}$, with $G_{\text{Nyq}} > 0$, $G_{\text{DC}} > 0$ and $G_{\text{Nyq}} \neq G_{\text{DC}}$. Then (1.8) yields the pair of equations

$$G_{\text{Nyq}} = |\gamma| \frac{|1+\beta|}{1+\alpha}, \quad G_{\text{DC}} = |\gamma| \frac{|1-\beta|}{1-\alpha}. \tag{1.58}$$

where we allow $\gamma < 0$. Assume for the moment that $|\beta| \leqslant 1$. Then the solution for β is

$$\beta = \frac{G_{\text{Nyq}} - G_{\text{DC}} + (G_{\text{Nyq}} + G_{\text{DC}})\alpha}{G_{\text{Nyq}} + G_{\text{DC}} + (G_{\text{Nyq}} - G_{\text{DC}})\alpha}. \tag{1.59}$$

We can also write (1.58) as

$$\frac{1 \pm \alpha}{1 \pm \beta} = \frac{G_{\text{Nyq}} + G_{\text{DC}} + (G_{\text{Nyq}} - G_{\text{DC}})\alpha}{2G_{\pm}} \tag{1.60}$$

where $G_+ = G_{\text{Nyq}}$ and $G_- = G_{\text{DC}}$. Now consider the geometric mean

$$G_{\text{avg}} = \sqrt{G_{\text{Nyq}} G_{\text{DC}}}. \tag{1.61}$$

This will occur at a frequency ω_{avg}, analogous to the cutoff frequency of the low-pass and high-pass cases. Substituting (1.60), (1.61) and either solution for $|\gamma|$ from (1.58) into (1.8) yields

$$4G_{\text{avg}}^2(1 - 2k\alpha + \alpha^2) = \big(G_{\text{Nyq}} + G_{\text{DC}} + (G_{\text{Nyq}} - G_{\text{DC}})\alpha\big)^2(1 - 2k\beta + \beta^2) \tag{1.62}$$

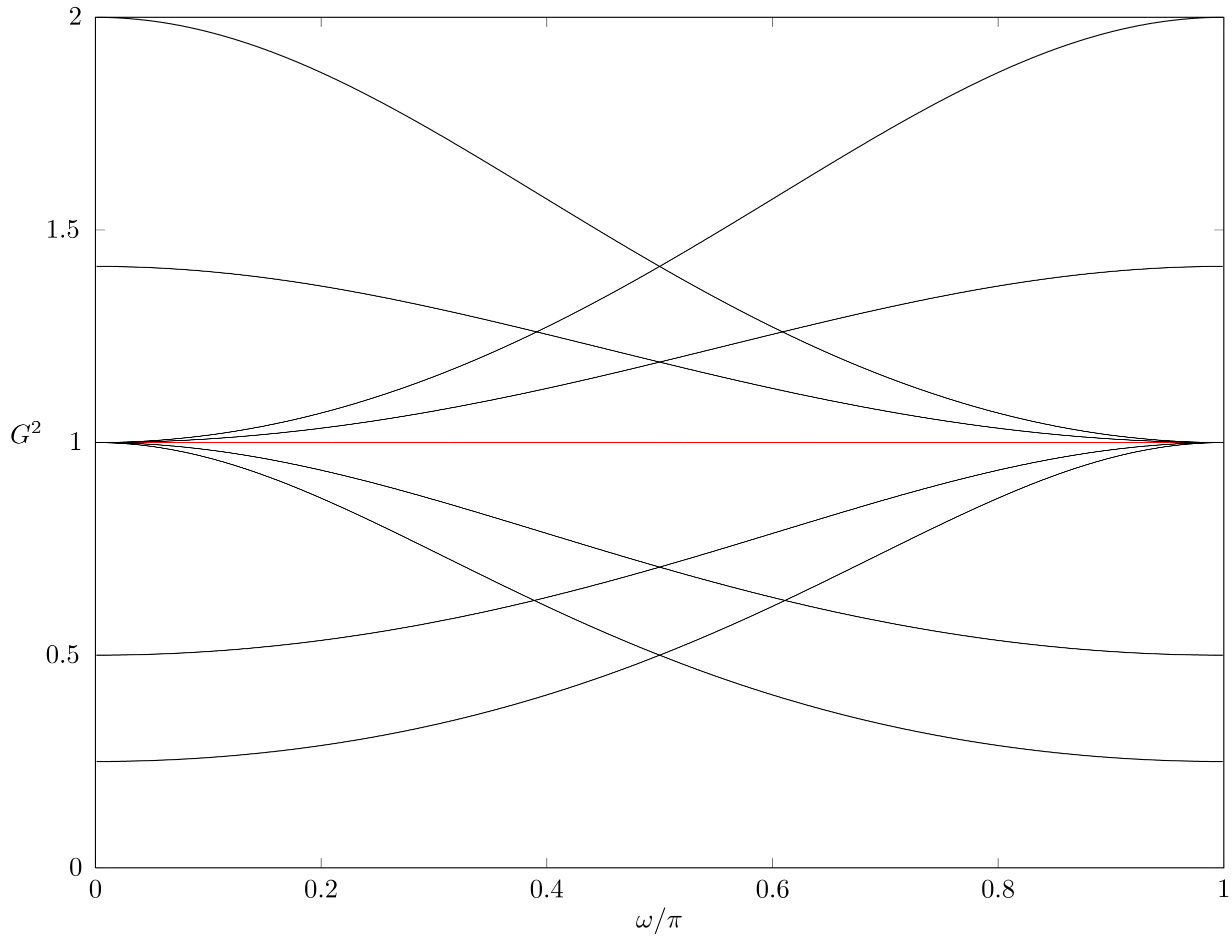

Figure 1.9: First-order shelf gain, $\omega_{\text{avg}}/\pi = 0.5$: Low-shelf, $G_{\text{DC}} = 0.5$, $\sqrt{0.5}$, $\sqrt[4]{2}$, $\sqrt{2}$;
High-shelf, $G_{\text{Nyq}} = 0.5$, $\sqrt{0.5}$, $\sqrt[4]{2}$, $\sqrt{2}$

where $k = \cos\omega_{\text{avg}}$. Substituting (1.59) into (1.62) yields the quadratic

$$\big(G_{\text{DC}} - G_{\text{Nyq}} + (G_{\text{DC}} + G_{\text{Nyq}})k\big)(\alpha^2 + 1) - 2\big(G_{\text{DC}} + G_{\text{Nyq}} + (G_{\text{DC}} - G_{\text{Nyq}})k\big)\alpha = 0. \tag{1.63}$$

with solution

$$\alpha = \frac{G_{\text{DC}} + G_{\text{Nyq}} + (G_{\text{DC}} - G_{\text{Nyq}})k - 2G_{\text{avg}}\sqrt{1-k^2}}{G_{\text{DC}} - G_{\text{Nyq}} + (G_{\text{DC}} + G_{\text{Nyq}})k} \tag{1.64}$$

choosing the appropriate root to ensure $|\alpha| < 1$ for stability. If instead $|\beta| \geqslant 1$, then the solution for β is

$$\beta = \frac{G_{\text{Nyq}} + G_{\text{DC}} + (G_{\text{Nyq}} - G_{\text{DC}})\alpha}{G_{\text{Nyq}} - G_{\text{DC}} + (G_{\text{Nyq}} + G_{\text{DC}})\alpha} \tag{1.65}$$

(reciprocal), with α again given by (1.64). For $|\gamma|$, either equation in (1.58) can be used (both yield the same value). The sign of γ is selected depending on whether the phase should be zero at DC or Nyquist.

1.3.4 All-pass

Here we will construct an *all-pass* filter, which lets all frequencies through with no attenuation or boost. This allows the phase response to be adjusted without affecting the amplitude response. To achieve this, we set

$$\beta = \frac{1}{\alpha}. \tag{1.66}$$

Then (1.8) implies

$$G(\omega) = \frac{|\gamma|}{|\alpha|} \tag{1.67}$$

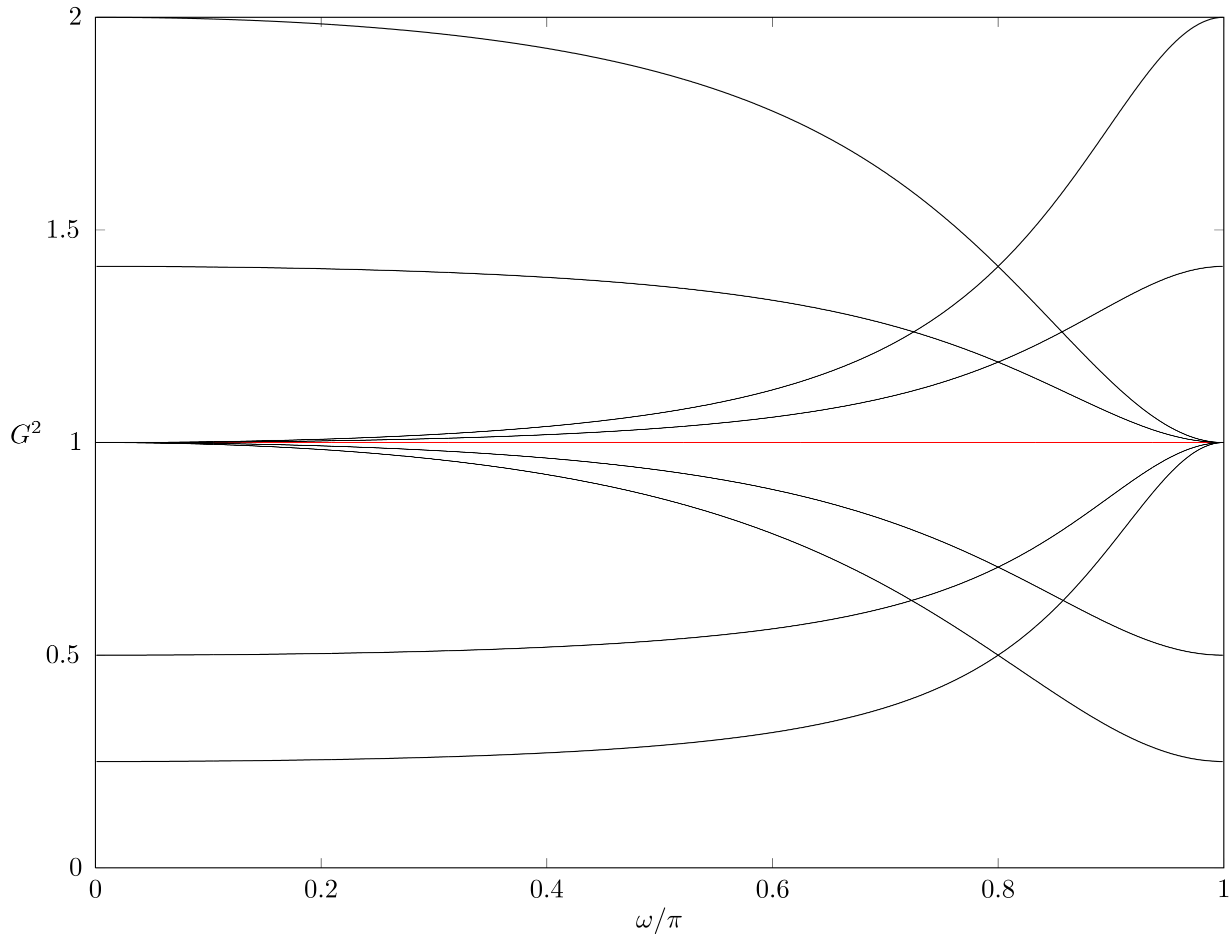

Figure 1.10: First-order shelf gain, $\omega_{\text{avg}}/\pi = 0.8$: Low-shelf, $G_{\text{DC}} = 0.5$, $\sqrt{0.5}$, $\sqrt[4]{2}$, $\sqrt{2}$; High-shelf, $G_{\text{Nyq}} = 0.5$, $\sqrt{0.5}$, $\sqrt[4]{2}$, $\sqrt{2}$

for all ω. Setting $G(\omega) = 1$ then implies

$$\gamma = s_{\pm}|\alpha|. \tag{1.68}$$

where $s_{\pm} = \pm 1$, allowing $\gamma < 0$ in the present case. The phase simplifies to

$$\Theta(\omega) = \arg(s_{\pm}|\alpha|) + \tan^{-1}\left\{\frac{(1-\alpha^2)\sin\omega}{2\alpha - (1+\alpha^2)\cos\omega}\right\}. \tag{1.69}$$

Consider the critical points of the derivative of (1.69):

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\omega} = \frac{(1-\alpha^2)\big(2\alpha\cos\omega - (1+\alpha^2)\big)}{(1+\alpha^2)^2 - 4\alpha(1+\alpha^2)\cos\omega + 4\alpha^2\cos^2\omega} \tag{1.70}$$

$$= 0. \tag{1.71}$$

Assuming a non-zero denominator, and rejecting $\alpha = \pm 1$, we require $2\alpha\cos\omega - (1+\alpha^2) = 0$, which has no real solutions for ω. So $\Theta(\omega)$ is a (strictly) monotonic function of ω. Thus there is a frequency yielding quadrature, occurring when the denominator of the inverse tangent term in (1.69) is zero:

$$(\alpha^2 + 1)\cos\omega_{\text{avg}_\Theta} - 2\alpha = 0 \tag{1.72}$$

which is the same quadratic form as (1.54), with solution

$$\alpha = \frac{1 - \sin\omega_{\text{avg}_\Theta}}{\cos\omega_{\text{avg}_\Theta}}. \tag{1.73}$$

(These lines here to ensure all-pass figure is shown.)

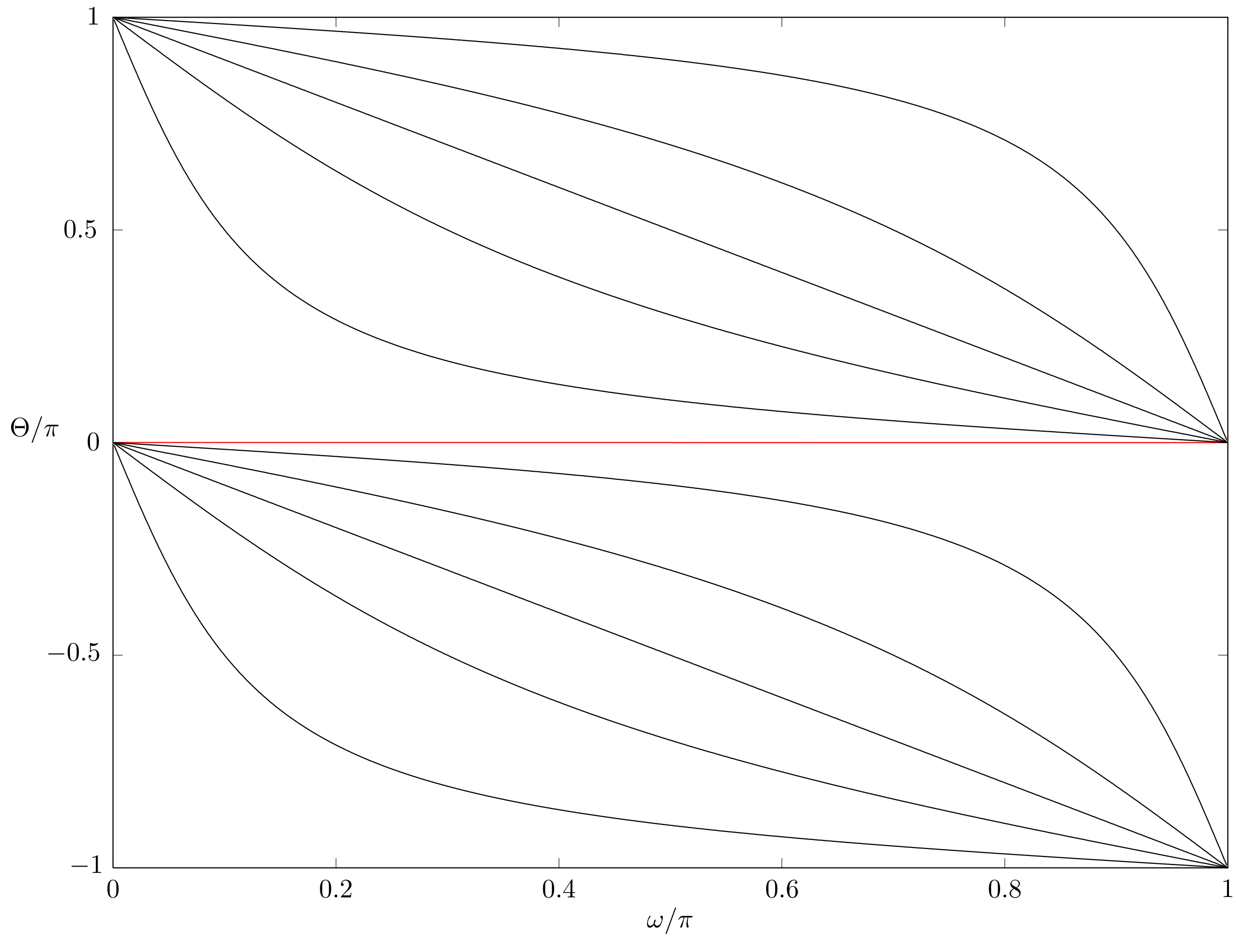

Figure 1.11: First-order all-pass: $s_{\pm} = 1$ (top), $\omega_{\text{avg}_\Theta}/\pi = 0.1, 0.3, 0.5, 0.7, 0.9$;
$s_{\pm} = -1$ (bottom), $\omega_{\text{avg}_\Theta}/\pi = 0.1, 0.3, 0.5, 0.7, 0.9$

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